Optimal stopping problem with reservation where reserving duration is determinable (Mathematical Modeling and Optimization under Uncertainty)

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Optimal stopping problem with reservation where reserving duration is determinable

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Abstract: Saito [10] dealt with an optimal stopping problem where each of offers appearing subsequently and randomly can be reserved as well as accepted or rejected but the term of validity of reservation is fixed to a certain number k. So, the model in this paper develops it into a determinable factor, that is, we can also determine how long an offer is reserved. The major finding is that almost all properties of the optimal decision rule in [10] hold for the current model, that is, an offer reserved during the search process must not be accepted prior to its maturity of reservation, however, it may be accepted on the maturity.

1. Introduction

In the situation where an offer among ones emerging subsequently and randomly has to be taken up to the deadline of decision making, a problem of determining which offer should be accepted when it should be accepted in order to maximize the expected profit is referred to as an optimal stopping problem. A point of the problem is what actions are taken to offers appearing up to when the offer to be accepted appears. If only rejection is allowed, future availabilities of past offers, or rejected offers, are out of our hands, and every rejected offer is automatically determined to become either certainly unavailable, or perfectly available, or uncertainly available in the future. These three types of the problem have so far been abundantly investigated, e.g.,[1][2][4][5][6][7][8][11].

If not only rejection but also reservation is allowed, by reserving an offer we leave chance to recall it in the future, that is, we get an ability to decide to make an offer either recallable or not in the future. Saito [9][10] dealt with such a model where every offer can be reserved in exchange for a certain compensation, and found that no reserved offer should be accepted while it remains available at the next point in time, in other words, you should not recall any reserved offer prior to the deadline or the maturity of its reservation (the concept of the maturity is considered only in [10]). However, *reserving duration*, or the term of validity of reservation, is assumed to be uncontrollable: in [9] it is limitless, or ∞, and in [10] it is restricted to k periods.

This drives us to the question what if the reserving duration becomes a controllable factor, and it is the question that we would like to resolve in this paper. More precisely, the purpose of this paper is to deal with the model where not only offers to be reserved but also their reserving durations are determinable, and to examine the properties of the optimal decision rule for the model. A major finding here is that almost all properties of the rules for models in [9] and [10] are inherited.

The model is precisely described in section 2. Preliminaries for the analyses are introduced in section 3, and the optimal equation of the model is formulated in section 4. In section 5 we analyze the properties of the optimal decision rule, and they are summarized in section 6.

2. Model

Up to the deadline you would like to accept as good an offer as possible among ones found at periodic intervals, where their values are i.i.d. random variables following a distribution function F such that $F(w) = 0$ for $w < a$, $0 < F(w) < 1$ for $a \le w < b$, $F(w) = 1$ for $b \le w$ with $0 \le a < b < \infty$, and the expectation is μ; and finding an offer at a point in time requires you to pay a *search cost* $s > 0$ at the preceding time.

On finding a new offer, or a *current offer*, you must inspect it and decide how to manage it. The choices available are accepting it, passing it up, and reserving it for l periods where the reserving duration l must be at most k. Reserving an offer w for l periods, of course, enables you to recall it for l periods from the time of reservation, but requires you to pay a *reserving cost* $r(w,l)$ which is positive and nondecreasing in w and l and continuous in w. Passing up the current offer deprives you of the right to recalling it forever.

Note here that although you have some recallable offers, the reserved offer to be recalled is only the most lucrative one. Let such reserved offer be called *leading offer*. Then, the actions which can be taken at each time except for the deadline are summarized as the following four: accepting the current offer and stopping the search (`AS`), reserving the current offer for l periods and continuing the search (`RC`), passing up the current offer and stopping the search by accepting the leading offer (`PS`), and passing up the current offer and continuing the search (`PC`), where `AS`, `RC`, `PS`, and `PC` represent the four decisions, respectively. Of course, at the deadline, only decisions `AS` and `PS` are permitted.

In the model, the value of time is considered by a discount factor β such as $0 < \beta \leq 1$, that is, the present value of q monetary units obtained at the next time is given by βq monetary units.

The objective here is to find an optimal decision rule that guides you to which action should be taken at each decision point so as to maximize the total expected discounted present net profit obtainable in the process ahead, that is, the expectation of the present discounted value of an accepted offer minus that of the amount of search costs and reserving costs paid over the periods from the present point in time to the termination of the search by accepting an offer. Consequently, at each time, the leading offer is the offer with the highest value of all reserved offers offers remaining recallable.

3. Preliminaries

Let us define a number α as $\alpha = -s + \beta\mu$. Let us define the following two functions:

$$\psi(x) = \int_a^b \max\{w, x\}dF(w), \tag{3.1}$$

$$\rho(x) = \beta\int_a^b \max\{w, x\}dF(w) - x - s \;=\; \beta\psi(x) - x - s. \tag{3.2}$$

Furthermore, let θ denote the root of equation $\rho(x) = 0$, if it exists.

Lemma 3.1

(a) $\psi(x) = \mu$ *for* $x \leq a$, $x < \psi(x)$ *for* $x < b$, *and* $\psi(x) = x$ *for* $b \leq x$.

(b) $\psi(x)$ *is continuous, convex, and nondecreasing in* x, *and strictly increasing in* $x \geq a$.

(c) $\rho(x)$ *is continuous, convex, and nonincreasing in* x, *and strictly decreasing in* $x \leq b$.

(d) θ *exists uniquely in* $[\alpha, b)$. *And,* $\alpha < \theta$ *if and only if* $a < \alpha$.

Proof: See Ikuta [3]. ∎

Consider a vector $\boldsymbol{p} = (p_1, p_2, \cdots, p_k) \in \mathbb{R}^k$. Let $\boldsymbol{p}_i$ denote the vector defined by removing the ith element p_i from $\boldsymbol{p}$, that is, $\boldsymbol{p}_i = (p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_k) \in \mathbb{R}^{k-1}$. For a vector $\boldsymbol{d}$, define indicator coefficient δ_i as $\delta_i = 1$ if $d_i \geq 0$ and $\delta_i = 0$ if $d_i < 0$. By use of a vector $\boldsymbol{d}$ and corresponding indicator coefficients δ_i's, let us define $\hat{p} = \max\{\delta_i p_i \mid i \in K\}$ and $\hat{p}_i = \max\{\delta_j p_j \mid j \in K\backslash\{i\}\}$ where $K = \{1, 2, \cdots, k\}$. Hence, we have $\hat{p} = \max\{\delta_i p_i, \hat{p}_i\}$ for each i.

4. Optimal Equation and Optimal Decision Rule

Let point in time t, simply referred to as *time* t later on, be equally spaced and numbered backward from the deadline $t = 0$, thus t also represents the number of periods remaining.

Suppose that we are at time t and the current offer of this time is w.

Let x_i denote the offer found i periods ago. Since any offer is not allowed to be reserved over k periods, reserving duration l for the current offer must satisfy $l \in K = \{1, 2, \cdots, k\}$. Accordingly, vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$ represents all offers each of which has a possibility of active reservation. We call the vector $\boldsymbol{x}$ *past vector*. We further define $\boldsymbol{y} \equiv \boldsymbol{x}_k = (x_1, x_2, \cdots, x_{k-1}) \in \mathbb{R}^{k-1}$ for notational convenience.

Let d_i denote the rest of duration of reservation of offer x_i. Although d_i is allowed to be negative for some technical reasons, the absolute value $|d_i|$ has no meaning if $d_i < 0$, and the inequality $d_i < 0$ means only that offer x_i is unavailable, that is, reservation of x_i has already expired or x_i was not reserved. Hence, offer x_i with $d_i = 0$ is on the maturity of reservation, offer x_j with $d_j = 1$ remains recallable at this time and next time, and so on. Let us call the vector $\boldsymbol{d} = (d_1, d_2, \cdots, d_k) \in \mathbb{R}^k$ *duration vector*, and define $\boldsymbol{c} \equiv \boldsymbol{d}_k = (d_1, d_2, \cdots, d_{k-1}) \in \mathbb{R}^{k-1}$ for convenience of later discussion.

Consequently, the leading offer of time t can be expressed as

$$\hat{x} = \max\{\delta_1 x_1,\ \delta_2 x_2,\ \cdots,\ \delta_k x_k\} \tag{4.1}$$

where δ_i's are indicator coefficients with respect to $\boldsymbol{d}$ such as $\delta_i = 1$ if $d_i \geq 0$ and $\delta_i = 0$ if $d_i < 0$.

If taking decision AS or PS, we will quit the search process by accepting the offer w or $\hat{x}$, respectively. If taking decision RC or PC, we are to continue the search. Then, since the offer x_k is surely unavailable at time $t-1$ whether it was reserved or not, the offers having chances to be recalled at time $t-1$ are expressed as

$$(w, \boldsymbol{y}) \equiv (w, x_1, x_2, \cdots, x_{k-1}) \in \mathbb{R}^k. \tag{4.2}$$

For each offer x_i in $\boldsymbol{x}$ of time t, the rest of its reserving duration d_i decreases one to time $t-1$. A similar thing must be considered for current offer w. Hence, if we pass up current offer w, the duration vector of time $t-1$ can be expressed as

$$(0, \boldsymbol{c}) - \mathbf{1} \equiv (-1, d_1 - 1, d_2 - 1, \cdots, d_{k-1} - 1) \in \mathbb{R}^k, \tag{4.3}$$

and if we reserve it for $l \in K$ periods, the duration vector of time $t-1$ can be expressed as

$$(l, \boldsymbol{c}) - \mathbf{1} \equiv (l - 1, d_1 - 1, d_2 - 1, \cdots, d_{k-1} - 1) \in \mathbb{R}^k. \tag{4.4}$$

Now, let us denote $u_t((w, \boldsymbol{x}); \boldsymbol{d})$ the maximum total expected present discounted net profit by starting time t on which we have past vector $\boldsymbol{x}$ with duration vector $\boldsymbol{d}$, and draw an offer w. Then,

$$u_t((w, \boldsymbol{x}); \boldsymbol{d}) = \max \left\{ \begin{array}{lcl} \texttt{AS} & : & w, \\ \texttt{RC} & : & -r(w, \eta^*) - s + \beta v_{t-1}((w, \boldsymbol{y}); (\eta^*, \boldsymbol{c}) - \mathbf{1}), \\ \texttt{PS} & : & \hat{x}, \\ \texttt{PC} & : & -s + \beta v_{t-1}((w, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1}) \end{array} \right\}, \quad t \geq 0, \tag{4.5}$$

where

$$v_t(\boldsymbol{x}; \boldsymbol{d}) = \int_a^b u_t((w, \boldsymbol{x}); \boldsymbol{d}) \mathrm{d}F(w), \quad t \geq 0; \qquad v_{-1}(\boldsymbol{x}; \boldsymbol{d}) = -\infty, \tag{4.6}$$

and

$$\eta^* = \eta_t(w, \boldsymbol{y}, \boldsymbol{d}) = \arg\max_{l \in K} \{-r(w, l) - s + \beta v_{t-1}((w, \boldsymbol{y}); (l, \boldsymbol{c}) - \mathbf{1})\}. \tag{4.7}$$

$v_t(\boldsymbol{x};\boldsymbol{d})$ is the expectation of $u_t((w,\boldsymbol{x});\boldsymbol{d})$ with respect to w, and $\eta_t(w,\boldsymbol{y},\boldsymbol{d})$ is the best reserving duration. Sometimes we shall write η^* instead of $\eta_t(w,\boldsymbol{y},\boldsymbol{d})$.

We immediately get $u_0((w,\boldsymbol{x});\boldsymbol{d}) = \max\{w,\hat{x}\}$, which shows that we must stop the process on the deadline by accepting either the current offer or the leading offer. Remember that the equality implies $v_0(\boldsymbol{x};\boldsymbol{d}) = \psi(\hat{x})$ for any $\boldsymbol{x}$ and $\boldsymbol{d}$ due to (3.1).

Lemma 4.1 *Suppose that $\boldsymbol{d} \in \mathbb{R}^k$ satisfies $d_i < 0$ for a certain i and that $\boldsymbol{x}^1 \in \mathbb{R}^k$ and $\boldsymbol{x}^2 \in \mathbb{R}^k$ satisfy $x_j^1 = x_j^2$ for $j \neq i$. Then, $v_t(\boldsymbol{x}^1;\boldsymbol{d}) = v_t(\boldsymbol{x}^2;\boldsymbol{d})$.*

Proof: Referring to (4.5), (4.6), and (3.1), we immediately get $v_0(\boldsymbol{x};\boldsymbol{d}') = \psi(\hat{x})$ for any $\boldsymbol{x}$ and $\boldsymbol{d}'$. Hence, since $\hat{x}^1 = \hat{x}^2$ due to (4.1) and $d_i < 0$, we acquire $v_0(\boldsymbol{x}^1;\boldsymbol{d}) = v_0(\boldsymbol{x}^2;\boldsymbol{d})$. We have thus confirmed the assertion for $t = 0$ and all i.

Suppose the assertion holds for $t-1$ and all i. Let $\boldsymbol{p}^1 = (w,\boldsymbol{y}^1) \in \mathbb{R}^k$ and $\boldsymbol{p}^2 = (w,\boldsymbol{y}^2) \in \mathbb{R}^k$ where w is any. Furthermore, let $\boldsymbol{q} = (l,\boldsymbol{c}) - \boldsymbol{1} \in \mathbb{R}^k$ where l is any.

If $i = k$, since the condition $x_j^1 = x_j^2$ for $j \neq k$ means $\boldsymbol{y}^1 = \boldsymbol{y}^2$, we get $\boldsymbol{p}^1 = \boldsymbol{p}^2$, hence $v_{t-1}(\boldsymbol{p}^1;\boldsymbol{q}) = v_{t-1}(\boldsymbol{p}^2;\boldsymbol{q})$. Next, fix $i < k$. By definition of $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, we know $p_j^1 = p_j^2$ for $j \neq i+1$. Since $d_i < 0$, we have $q_{i+1} = d_i - 1 < 0$. Hence, by induction assumption we get $v_{t-1}(\boldsymbol{p}^1;\boldsymbol{q}) = v_{t-1}(\boldsymbol{p}^2;\boldsymbol{q})$.

Note that $\hat{x}^1 = \hat{x}^2$ due to $d_i < 0$, thus $\delta_i = 0$, and (4.1).

From these above and (4.5), we obtain $u_t((w,\boldsymbol{x}^1);\boldsymbol{d}) = u_t((w,\boldsymbol{x}^2);\boldsymbol{d})$, thus $v_t(\boldsymbol{x}^1;\boldsymbol{d}) = v_t(\boldsymbol{x}^2;\boldsymbol{d})$. Hence, by induction we complete the proof. ∎

It is intuitively clear that offers unrecallable have nothing to do with the search process. Lemma 4.1 assures it mathematically. Here, we redefine $u_t((w,\boldsymbol{x});\boldsymbol{d})$ from (4.5) to

$$u_t((w,\boldsymbol{x});\boldsymbol{d}) = \max\left\{\begin{array}{lcl} \texttt{AS} & : & w, \\ \texttt{RC} & : & -r(w,\eta^*) - s + \beta v_{t-1}((w,\boldsymbol{y});(\eta^*,\boldsymbol{c}) - \boldsymbol{1}), \\ \texttt{PS} & : & \hat{x}, \\ \texttt{PC} & : & -s + \beta v_{t-1}((0,\boldsymbol{y});(0,\boldsymbol{c}) - \boldsymbol{1}) \end{array}\right\}, \quad t \geq 0, \tag{4.8}$$

where only the expression for PC is different from that in (4.5).

Let us define the two functions $z_t^o(\boldsymbol{x};\boldsymbol{d})$ and $z_t^r((w,\boldsymbol{y});\boldsymbol{d})$ as follows:

$$z_t^o(\boldsymbol{x};\boldsymbol{d}) = \max\{\hat{x},\ -s + \beta v_{t-1}((0,\boldsymbol{y});(0,\boldsymbol{c}) - \boldsymbol{1})\}, \qquad t \geq 0, \tag{4.9}$$

$$z_t^r((w,\boldsymbol{y});\boldsymbol{d}) = \max\{w,\ -r(w,\eta^*) - s + \beta v_{t-1}((w,\boldsymbol{y});(\eta^*,\boldsymbol{c}) - \boldsymbol{1})\}, \qquad t \geq 0. \tag{4.10}$$

Clearly, $z_0^o(\boldsymbol{x};\boldsymbol{d}) = \hat{x}$ and $z_0^r((w,\boldsymbol{y});\boldsymbol{d}) = w$. From (4.8) we know that $z_t^o(\boldsymbol{x};\boldsymbol{d})$ and $z_t^r((w,\boldsymbol{y});\boldsymbol{d})$ stand for the total expected present discounted net profits attainable by passing up or not passing up the current offer w, respectively, at time t, and then following the optimal decision rule. Therefore, the set of current offers which should be accepted or reserved can be denoted by

$$W_t(\boldsymbol{x},\boldsymbol{d}) = \{w \mid z_t^o(\boldsymbol{x};\boldsymbol{d}) \leq z_t^r((w,\boldsymbol{y});\boldsymbol{d})\} \subseteq \mathbb{R}, \qquad t \geq 0. \tag{4.11}$$

By using these, we obtain

$$u_t((w,\boldsymbol{x});\boldsymbol{d}) = \max\{z_t^o(\boldsymbol{x};\boldsymbol{d}), z_t^r((w,\boldsymbol{y});\boldsymbol{d})\}, \tag{4.12}$$

and furthermore,

$$v_t(\boldsymbol{x};\boldsymbol{d}) = \int_a^b \max\{z_t^o(\boldsymbol{x};\boldsymbol{d}), z_t^r((w,\boldsymbol{y});\boldsymbol{d})\}\mathrm{d}F(w) \tag{4.13}$$

$$= \int_{W_t(\boldsymbol{x};\boldsymbol{d})} z_t^r((w,\boldsymbol{y});\boldsymbol{d})\mathrm{d}F(w) + \int_{W_t(\boldsymbol{x};\boldsymbol{d})^c} z_t^o(\boldsymbol{x};\boldsymbol{d})\mathrm{d}F(w). \tag{4.14}$$

For each i, given any $\boldsymbol{x}_i \in \mathbb{R}^{k-1}$ and $\boldsymbol{d} \in \mathbb{R}^k$, define a function g_t^i of ξ as

$$g_t^i(\xi; \boldsymbol{x}_i, \boldsymbol{d}) = -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \boldsymbol{1}) - \delta_i \xi, \qquad i \leq k, \tag{4.15}$$

where $\boldsymbol{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$ and

$$\boldsymbol{y} = \begin{cases} (x_1, \cdots, x_{i-1}, \xi, x_{i+1}, \cdots, x_{k-1}), & i < k, \\ (x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{k-1}) = \boldsymbol{x}_k, & i = k, \end{cases}$$

and define $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$ as a root of $g_t^i(\xi; \boldsymbol{x}_i, \boldsymbol{d}) = 0$, if any.

Given any $\boldsymbol{y} \in \mathbb{R}^{k-1}$ and $\boldsymbol{d} \in \mathbb{R}^k$, define a function f_t of ξ as

$$f_t(\xi; \boldsymbol{y}, \boldsymbol{d}) = -r(\xi, \eta^*) - s + \beta v_{t-1}((\xi, \boldsymbol{y}); (\eta^*, \boldsymbol{c}) - \boldsymbol{1}) - \xi \tag{4.16}$$

where $\eta^* = \eta_t(\xi, \boldsymbol{y}, \boldsymbol{d})$, and define $\lambda_t(\boldsymbol{y}, \boldsymbol{d})$ as a root of $f_t(\xi; \boldsymbol{y}, \boldsymbol{d}) = 0$, if any.

Compare (4.15) and (4.16) with (4.8). Then, we find that $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$ represents the indifferent point of time t in terms of x_i between accepting the offer x_i and continuing the search under given $\boldsymbol{x}_i$ and $\boldsymbol{d}$. The meaning of $\lambda_t(\boldsymbol{y}, \boldsymbol{d})$ is a point of indifference of time t in terms of w between accepting current offer w and reserving it for η^* periods under given $\boldsymbol{y}$ and $\boldsymbol{d}$.

Therefore, the optimal decision rule can be described by using $W_t(\boldsymbol{x}, \boldsymbol{d})$, $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$, and $\lambda_t(\boldsymbol{y}, \boldsymbol{d})$.

5. Analysis

Lemma 5.1 *Given any $t \geq 0$ and any $\boldsymbol{d} \in \mathbb{R}^k$:*

(a) $v_t(\boldsymbol{x}; \boldsymbol{d})$ *is continuous, convex, and nondecreasing in* $\boldsymbol{x}$.

(b) $v_t(\boldsymbol{x}; \boldsymbol{d})$ *is nondecreasing in* $\boldsymbol{d}$.

(c) $v_t(\boldsymbol{x}; \boldsymbol{d})$ *is nondecreasing in t, that is,* $v_{t-1}(\boldsymbol{x}; \boldsymbol{d}) \leq v_t(\boldsymbol{x}; \boldsymbol{d})$ *for* $t \geq 0$.

(d) $\mu \leq v_t(\boldsymbol{x}; \boldsymbol{d})$ *for any* $\boldsymbol{x}$, $\hat{x} < v_t(\boldsymbol{x}; \boldsymbol{d})$ *while* $\hat{x} < b$, *and* $v_t(\boldsymbol{x}; \boldsymbol{d}) = b$ *if* $\hat{x} = b$.

(e) $\beta v_t((0, \boldsymbol{y}); \boldsymbol{d}) - x_i$ *is strictly decreasing in x_i for any i and* $\boldsymbol{x}_i \in \mathbb{R}^{k-1}$.

(f) $\beta v_t((w, \boldsymbol{y}); \boldsymbol{d}) - w$ *is strictly decreasing in w for any* $\boldsymbol{y} \in \mathbb{R}^{k-1}$.

Proof: (a) Since $u_0((w, \boldsymbol{x}); \boldsymbol{d}) = \max\{w, \hat{x}\}$, we deduce $v_0(\boldsymbol{x}; \boldsymbol{d}) = \psi(\hat{x})$ by (3.1), thus the assertion holds true for $t = 0$ due to Lemma 3.1(b).

Assume the assertion to be true for $t-1$. The expressions for PS and PC in (4.8) hold the three properties, respectively, by virtue of arbitrariness of $\boldsymbol{d}$ in the assumption. So also does the expression for AS because it is independent of $\boldsymbol{x}$. As for RC, we shall first show nondecreasing property. The assumption indicates $-r(w,l) - s + \beta v_{t-1}((w, \boldsymbol{y}^1); (l, \boldsymbol{c}) - \boldsymbol{1}) \leq -r(w,l) - s + \beta v_{t-1}((w, \boldsymbol{y}^2); (l, \boldsymbol{c}) - \boldsymbol{1})$ for any w, l, $\boldsymbol{d}$, $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ such that $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$. Hence, $-r(w, \eta^1) - s + \beta v_{t-1}((w, \boldsymbol{y}^2); (\eta^1, \boldsymbol{c}) - \boldsymbol{1}) = \max_{l \in K}\{-r(w,l) - s + \beta v_{t-1}((w, \boldsymbol{y}^1); (l, \boldsymbol{c}) - \boldsymbol{1})\} \leq \max_{l \in K}\{-r(w,l) - s + \beta v_{t-1}((w, \boldsymbol{y}^2); (l, \boldsymbol{c}) - \boldsymbol{1})\} = -r(w, \eta^2) - s + \beta v_{t-1}((w, \boldsymbol{y}^2); (\eta^2, \boldsymbol{c}) - \boldsymbol{1})$ where $\eta^1 = \eta_t(w, \boldsymbol{y}^1, \boldsymbol{d})$ and $\eta^2 = \eta_t(w, \boldsymbol{y}^2, \boldsymbol{d})$. Continuity and convexity can be shown in a similar way. Thereby, since all four expressions in braces of (4.8) satisfy these three properties, so also does $v_t(\boldsymbol{x}, \boldsymbol{d})$. We have thus completed the proof.

(b) Choose $\boldsymbol{d}^1 \in \mathbb{R}^k$ and $\boldsymbol{d}^2 \in \mathbb{R}^k$ so that $\boldsymbol{d}^1 \leq \boldsymbol{d}^2$. Define δ_i^m as $\delta_i^m = 1$ if $d_i^m \geq 0$ and $\delta_i^m = 0$ if $d_i^m < 0$ for $m = 1, 2$. Then, the number of δ_i^1's with $\delta_i^1 = 1$ is less than or equal to that of δ_i^2's, which implies $\max\{\delta_i^1 x_i\} \leq \max\{\delta_i^2 x_i\}$ for any $\boldsymbol{x}$, thus $\hat{x}$ is nondecreasing in $\boldsymbol{d}$. Using the fact and imitating the way of proof of (a) completes the proof.

(c) It follows from (4.8) that $u_0((w, \boldsymbol{x}); \boldsymbol{d}) = \max\{w, \hat{x}\} \leq u_1((w, \boldsymbol{x}); \boldsymbol{d})$, implying $v_0(\boldsymbol{x}; \boldsymbol{d}) \leq v_1(\boldsymbol{x}; \boldsymbol{d})$. Hence, $v_0(\boldsymbol{x}; (l, \boldsymbol{c}) - \boldsymbol{1}) \leq v_1(\boldsymbol{x}; (l, \boldsymbol{c}) - \boldsymbol{1})$ for any w and l even when $l = 0$, thus $u_1((w, \boldsymbol{x}); \boldsymbol{d}) \leq u_2((w, \boldsymbol{x}); \boldsymbol{d})$. By repeating this argument we can obtain the assertion.

(d) First, due to (4.8) and (4.6) we get $v_t(\boldsymbol{x};\boldsymbol{d}) \geq \int_a^b w dF(w) = \mu$. Secondly, if $\hat{x} < b$, then $\hat{x} < \psi(\hat{x}) = v_0(\boldsymbol{x};\boldsymbol{d}) \leq v_1(\boldsymbol{x};\boldsymbol{d}) \leq \cdots$ from assertion (a) and Lemma 3.1(a). Finally, if $\hat{x} = b$, then $v_0(\boldsymbol{x};\boldsymbol{d}) = \psi(b) = b$ from Lemma 3.1(a). Assume the assertion holds true for $t-1$ and let $\boldsymbol{b} = (b, \cdots, b) \in \mathbb{R}^{k-1}$. If $\hat{x} = b$, then $\hat{y} \leq b$, thus $\boldsymbol{y} \leq \boldsymbol{b}$, hence $v_{t-1}((w,\boldsymbol{y});\boldsymbol{d}) \leq v_{t-1}((w,\boldsymbol{b});\boldsymbol{d}) = b$ for any w and $\boldsymbol{d}$ due to assertion (a) and the induction assumption. Hence, if $\hat{x} = b$, since $\boldsymbol{d}$ is arbitrary, $u_t((w,\boldsymbol{x});\boldsymbol{d}) = b$ due to (4.8), thus $v_t(\boldsymbol{x};\boldsymbol{d}) = b$ from (4.6). Accordingly, the final part of the assertion holds true for every t.

(e) First, suppose $i = k$. Then, since $\boldsymbol{y} \in \mathbb{R}^{k-1}$ is independent of x_k, so also is $v_t((0,\boldsymbol{y});\boldsymbol{d})$, thus $\beta v_t((0,\boldsymbol{y});\boldsymbol{d}) - x_k$ is strictly decreasing in x_k for any $\boldsymbol{x}_k\ (= \boldsymbol{y}) \in \mathbb{R}^{k-1}$.

Next, suppose $i < k$ and $\hat{y}_i = b$. Then, for any x_i we have $\hat{y} = \max\{x_i, \hat{y}_i\} = b$, implying $v_t((0,\boldsymbol{y});\boldsymbol{d}) = b$ due to (d). Hence the assertion holds true if $\hat{y}_i = b$.

Finally, suppose $i < k$ and $\hat{y}_i < b$. Choose k-dimensional vectors $\boldsymbol{x}^1$, $\boldsymbol{x}^2$, and $\boldsymbol{x}^b$ so that $x_i^1 < x_i^2 < x_i^b = b$ and $x_j^1 = x_j^2 = x_j^b < b$ for $j \neq i, k$. Then, $\hat{y}_i^1 = \hat{y}_i^2 = \hat{y}_i^b < b$ and $\hat{y}^1 \leq \hat{y}^2 < \hat{y}^b = b$. Hence, by (d) we get $v_t((0,\boldsymbol{y}^b);\boldsymbol{d}) = b$ and $x_i^1 \leq \hat{y}^1 < v_t((0,\boldsymbol{y}^1);\boldsymbol{d})$. Accordingly, since $v_t((0,\boldsymbol{y});\boldsymbol{d})$ is convex in x_i, we obtain

$$\beta\frac{v_t((0,\boldsymbol{y}^2);\boldsymbol{d}) - v_t((0,\boldsymbol{y}^1);\boldsymbol{d})}{x_i^2 - x_i^1} \leq \beta\frac{v_t((0,\boldsymbol{y}^b);\boldsymbol{d}) - v_t((0,\boldsymbol{y}^1);\boldsymbol{d})}{b - x_i^1} < \beta\frac{b - x_i^1}{b - x_i^1} \leq 1,$$

from which $\beta v_t((0,\boldsymbol{y}^1);\boldsymbol{d}) - x_i^1 > \beta v_t((0,\boldsymbol{y}^2);\boldsymbol{d}) - x_i^2$. Thus the proof is completed.

(f) The assertion can be verified in a like manner to the proof of (e). ∎

Theorem 5.2

(a) *If $\boldsymbol{x} \in \mathbb{R}^k$ and $\boldsymbol{d} \in \mathbb{R}^k$ are such that $\theta \leq \hat{x}$ where θ is the root of equation $\rho(x) = 0$, then $u_t((w,\boldsymbol{x});\boldsymbol{d}) = \max\{w, \hat{x}\}$ and thus $v_t(\boldsymbol{x};\boldsymbol{d}) = \psi(\hat{x})$.*

(b) *If $\alpha \leq a$, then $u_t((w,\boldsymbol{x});\boldsymbol{d}) = \max\{w, \hat{x}\}$ and $v_t(\boldsymbol{x};\boldsymbol{d}) = \psi(\hat{x})$ for any $\boldsymbol{x}$ and $\boldsymbol{d}$.*

Proof: (a) It is clear for $t = 0$ due to (4.8), (4.6), and (3.1). Choose $\boldsymbol{x}$ and $\boldsymbol{d}$ so that $\theta \leq \hat{x}$, and suppose the assertion to be true for $t-1$. Let δ denote an indicator about $l \in K \cup \{0\}$ such that $\delta = 1$ if $l \in K$ and $\delta = 0$ if $l = 0$. Then, since $\theta \leq \hat{x}$, for any w and l we have $\theta \leq \max\{\delta w, \hat{x}\}$. Thus, Lemma 5.1(b,a) and the induction assumption lead us to

$$v_{t-1}((w,\boldsymbol{y});(l,\boldsymbol{c}) - \boldsymbol{1}) \leq v_{t-1}((w,\boldsymbol{y});(l,\boldsymbol{k})) \leq v_{t-1}((w,\hat{x},\cdots,\hat{x});(l,\boldsymbol{k})) = \psi(\max\{\delta w, \hat{x}\}) \quad (5.1)$$

where $\boldsymbol{k} = (k, \cdots, k) \in \mathbb{R}^{k-1}$. Since $\theta \leq \max\{\delta w, \hat{x}\}$, Lemma 3.1(c) produces $0 = \rho(\theta) \geq \rho(\max\{\delta w, \hat{x}\}) = -s + \beta\psi(\max\{\delta w, \hat{x}\}) - \max\{\delta w, \hat{x}\}$. From this inequality and (5.1) we acquire

$$-s + \beta v_{t-1}((w,\boldsymbol{y});(l,\boldsymbol{c}) - \boldsymbol{1}) \leq -s + \beta\psi(\max\{\delta w, \hat{x}\}) \leq \max\{\delta w, \hat{x}\}. \quad (5.2)$$

By considering $l = 0$, or $\delta = 0$ in (5.2) and referring to (4.9), we get $z_t^o(\boldsymbol{x};\boldsymbol{d}) = \hat{x}$.

If $w < \hat{x}$, by noting $r(w,l) > 0$ for any $l \in K$, it follows from (5.2) and (4.10) that $z_t^r((w,\boldsymbol{y});\boldsymbol{d}) < \max\{\hat{x}, \hat{x}\} = \hat{x}$. Similarly, if $\hat{x} \leq w$, we have $z_t^r((w,\boldsymbol{y});\boldsymbol{d}) = w$.

From above we conclude from (4.12) that if $\theta \leq \hat{x}$, then $u_t((w,\boldsymbol{x});\boldsymbol{d}) = \max\{z_t^r((w,\boldsymbol{y});\boldsymbol{d}), \hat{x}\} = \max\{w, \hat{x}\}$, hence $v_t(\boldsymbol{x};\boldsymbol{d}) = \psi(\hat{x})$. We have thus completed the induction.

(b) Suppose $\alpha \leq a$. Then, $\theta = \alpha$ due to Lemma 3.1(d), thus $\theta \leq a$.

Suppose $\boldsymbol{d}$ satisfies $d_i \geq 0$ at least one i. Then, for any $\boldsymbol{x}$, since $x_i \geq a$ for all i, by (4.1) we get $a \leq \hat{x}$, thus $\theta \leq \hat{x}$. Thereby $u_t((w,\boldsymbol{x});\boldsymbol{d}) = \max\{w, \hat{x}\}$ from assertion (a).

Suppose $\boldsymbol{d}$ satisfies $d_i < 0$ for all i, thus $\boldsymbol{c} < \boldsymbol{0}$ and $\hat{x} = 0 \leq a \leq w$ for any $\boldsymbol{x}$ and w. Consider a vector $(l,\boldsymbol{c}) - \boldsymbol{1} \in \mathbb{R}^k$ with given any l. Its all elements except for the first one are surely negative because of $\boldsymbol{c} < \mathbf{0}$. Hence, due to Lemmas 4.1 and 5.1(b) and assertion (a) we obtain

$$\begin{aligned} v_{t-1}((w,\boldsymbol{y});(l,\boldsymbol{c})-\mathbf{1}) &= v_{t-1}((w,\theta,\cdots,\theta);(l,\boldsymbol{c})-\mathbf{1}) \\ &\le v_{t-1}((w,\theta,\cdots,\theta);(l-1,0,\cdots,0)) = \psi(\max\{\delta w,\hat{x}\}) = \psi(\delta w) \quad (5.3)\end{aligned}$$

where δ is such that $\delta = 1$ if $l \in K$ and $\delta = 0$ if $l = 0$.

If $\delta = 0$, for any w we get $-s+\beta v_{t-1}((w,\boldsymbol{y});(l,\boldsymbol{c})-\mathbf{1}) \le -s+\beta\psi(0) = -s+\beta\mu = \alpha \le a \le w$. If $\delta = 1$, since $\theta \le a \le \delta w = w$ for any w, it follows from Lemma 3.1(c) that $0 \ge \rho(\delta w)$. This relation and (5.3) provide $-s+\beta v_{t-1}((w,\boldsymbol{y});(l,\boldsymbol{c})-\mathbf{1}) \le -s+\beta\psi(\delta w) \le \delta w = w$, from which we immediately find that expressions for RC and PC in (4.8) are both less than or equal to w. Hence, $u_t((w,\boldsymbol{x});\boldsymbol{d}) = w = \max\{w,\hat{x}\}$ holds when $\boldsymbol{d} < \mathbf{0}$. As a result, we get $v_t(\boldsymbol{x};\boldsymbol{d}) = \psi(\hat{x})$ for any $\boldsymbol{x}$ and $\boldsymbol{d}$ when $\alpha \le a$. ∎

Theorem 5.2(a) provides the following rule:

Optimal decision rule I: Suppose that you are at time t on which you have a recallable offer $\hat{x}$ with value more then or equal to θ and draw an offer w. Then, the optimal choice is stopping the search process by accepting the more lucrative one between current offer w and leading offer $\hat{x}$.

By noting that both α and a are independent of t, Theorem 5.2(b) produces the following:

Optimal decision rule II: Suppose a condition of the search process is $\alpha \le a$. Then, you should stop the search process by accepting the more lucrative one between current offer w and leading offer $\hat{x}$ as soon as you start the process.

By considering the meanings of α and a, the rule II tells us that if we must search offers in situation where the expected discounted profit from one more search (α) is not over the lowest value of offers (a), it is unprofitable to engage in the process. Accordingly, to exclude such a trivial case, we hereafter assume

$$a < \alpha \ (= -s+\beta\mu) \quad (5.4)$$

Lemma 5.3 *For $t \ge 1$:*

(a) $\theta_t^i(\boldsymbol{x}_i,\boldsymbol{d})$ *exists uniquely with* $\alpha \le \theta_t^i(\boldsymbol{x}_i,\boldsymbol{d}) < b$ *for each* i *such that* $d_i \ge 0$. $\theta_t^i(\boldsymbol{x}_i,\boldsymbol{d})$ *does not exist for* i *such that* $d_i < 0$.

(b) $\lambda_t(\boldsymbol{y},\boldsymbol{d})$ *exists uniquely with* $\lambda_t(\boldsymbol{y},\boldsymbol{d}) < b$.

Proof: (a) Suppose first $\boldsymbol{d}$ is such that $d_i \ge 0$, thus $\delta_i = 1$. Then, it follows from (4.15) and Lemma 5.1(e) that $g_t^i(\xi;\boldsymbol{x}_i,\boldsymbol{d})$ is strictly decreasing in ξ for any $\boldsymbol{x}_i$. Furthermore, from Lemma 5.1(d) we get $g_t^i(\alpha;\boldsymbol{x}_i,\boldsymbol{d}) \ge -s+\beta\mu-\alpha = 0$ and $g_t^i(b;\boldsymbol{x}_i,\boldsymbol{d}) = -s+(\beta-1)b < 0$. Hence, equation $g_t^i(\alpha;\boldsymbol{x}_i,\boldsymbol{d}) = 0$ has a unique root in $[\alpha,b)$.

Next, suppose $\boldsymbol{d}$ is such that $d_i < 0$, thus $\delta_i = 0$. Then, it follows from (5.4) and Lemma 5.1(d) that $g_t^i(\xi;\boldsymbol{x}_i,\boldsymbol{d}) \ge -s+\beta\mu > a \ge 0$ for any ξ. Hence, there is no root of equation $g_t^i(\xi;\boldsymbol{x}_i,\boldsymbol{d}) = 0$.

(b) By almost the same method as in the previous proof, we get $\lim_{\xi\to-\infty} f_t(\xi;\boldsymbol{y},\boldsymbol{d}) = \infty$ and $f_t(b;\boldsymbol{y},\boldsymbol{d}) < 0$, and conclude the assertion to be true. ∎

As seen in Lemma 5.3(a), we have cases such that $\theta_t^i(\boldsymbol{x}_i,\boldsymbol{d})$ does not exist. In such cases we define $\theta_t^i(\boldsymbol{x}_i,\boldsymbol{d}) = \infty$ for convenience. Furthermore, for $t = 0$, we define $\theta_0^i(\boldsymbol{x}_i,\boldsymbol{d}) = -\infty$ and $\lambda_0(\boldsymbol{y},\boldsymbol{d}) = -\infty$, respectively. Then, we obtain the following corollary.

Corollary 5.4

(a) *For any* $\boldsymbol{d} \in \mathbb{R}^k$, i *and* $\boldsymbol{x}_i \in \mathbb{R}^{k-1}$:

(1) *if* $x_i < \theta_t^i(\boldsymbol{x}_i,\boldsymbol{d})$, *then* $\delta_i x_i < -s+\beta v_{t-1}((0,\boldsymbol{y});(0,\boldsymbol{c})-\mathbf{1})$,

(2) *if* $x_i = \theta_t^i(\boldsymbol{x}_i,\boldsymbol{d})$, *then* $\delta_i x_i = -s+\beta v_{t-1}((0,\boldsymbol{y});(0,\boldsymbol{c})-\mathbf{1})$,

(3) *if* $x_i > \theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$, *then* $\delta_i x_i > -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1})$.

(b) *For any* $\boldsymbol{d} \in \mathbb{R}^k$ *and* $\boldsymbol{y} \in \mathbb{R}^{k-1}$:

(1) *if* $w < \lambda_t(\boldsymbol{y}, \boldsymbol{d})$, *then* $w < -r(w, \eta^*) - s + \beta v_{t-1}((w, \boldsymbol{y}); (\eta^*, \boldsymbol{c}) - \mathbf{1})$,

(2) *if* $w = \lambda_t(\boldsymbol{y}, \boldsymbol{d})$, *then* $w = -r(w, \eta^*) - s + \beta v_{t-1}((w, \boldsymbol{y}); (\eta^*, \boldsymbol{c}) - \mathbf{1})$,

(3) *if* $w > \lambda_t(\boldsymbol{y}, \boldsymbol{d})$, *then* $w > -r(w, \eta^*) - s + \beta v_{t-1}((w, \boldsymbol{y}); (\eta^*, \boldsymbol{c}) - \mathbf{1})$.

Proof: (a) Let $t \geq 1$. Suppose $\boldsymbol{d}$ is such that $d_i \geq 0$, thus $\delta_i = 1$. Then, $g_t^i(\xi; \boldsymbol{x}_i, \boldsymbol{d})$ is strictly decreasing in ξ for any $\boldsymbol{x}_i$ as shown in the proof of Lemma 5.3(a). Hence, the assertion holds true in this case. If $\boldsymbol{d}$ is such that $d_i < 0$, thus $\delta_i = 0$, we know $g_t^i(\xi; \boldsymbol{x}_i, \boldsymbol{d}) > 0$, that is, $\delta_i \xi < -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1})$ for any ξ with any $\boldsymbol{x}_i$. In this case, however, we have defined $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) = \infty$, thus we regard the previous inequality holds while $\xi < \theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$.

In the case of $t = 0$, from (4.6) we have $\xi > -s + \beta v_{-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1})$ for any finite ξ. Since we have defined $\theta_0^i(\boldsymbol{x}_i, \boldsymbol{d}) = -\infty$, we can regard the previous inequality holds while $\xi > \theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$ with $t = 0$.

(b) Easy by almost the same method as in the previous proof. ∎

By use of Corollary 5.4 we find that if $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) < x_i$ for a certain i, then $-s+\beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1}) < \max\{\delta_1 x_1, \cdots, \delta_k x_k\} = \hat{x}$, thus $z_t^o(\boldsymbol{x}; \boldsymbol{d}) = \hat{x}$. If $x_i \leq \theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$ for all i, similarly we get $z_t^o(\boldsymbol{x}; \boldsymbol{d}) = -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1})$.

From this and (4.11) we can prescribe an optimal decision rule as follows:

Optimal decision rule III: Suppose that you are at time t on which you have past vector $\boldsymbol{x}$ with duration vector $\boldsymbol{d}$, and draw an offer w. Then, the optimal choices are:

(a) if $w \in W_t(\boldsymbol{x}, \boldsymbol{d})$, then:

- if $\lambda_t(\boldsymbol{y}, \boldsymbol{d}) < w$, AS (accept current offer w and stop the search)
- if $w \leq \lambda_t(\boldsymbol{y}, \boldsymbol{d})$, RC (reserve current offer w for $\eta_t(w, \boldsymbol{y}, \boldsymbol{d})$ periods and continue the search)

(b) if $w \notin W_t(\boldsymbol{x}, \boldsymbol{d})$, then:

- if $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) < x_i = \hat{x}$ for a certain i, PS (pass up current offer w and stop the search by accepting the leading offer $\hat{x}$)
- if $\hat{x} \leq \theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$ for all i, PC (pass up current offer w and continue the search)

Theorem 5.5

(a) $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$ *is nondecreasing in* $\boldsymbol{x}_i$ *and* $\boldsymbol{d}$, *and* $\lambda_t(\boldsymbol{y}, \boldsymbol{d})$ *is nondecreasing in* $\boldsymbol{y}$ *and* $\boldsymbol{d}$.

(b) *If* $\boldsymbol{x}_i \in \mathbb{R}^{k-1}$ *and* $\boldsymbol{d} \in \mathbb{R}^k$ *are such that* $\hat{x}_i \leq \theta$ *and* $d_i \geq 1$, *then* $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) = \theta$.

(c) *If* $\boldsymbol{x}_i \in \mathbb{R}^{k-1}$ *and* $\boldsymbol{d} \in \mathbb{R}^k$ *are such that* $\hat{x}_i \leq \theta$ *and* $d_i = 0$, *then* $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) \leq \theta$.

(d) *If* $\boldsymbol{y} \in \mathbb{R}^{k-1}$ *and* $\boldsymbol{d} \in \mathbb{R}^k$ *are such that* $\hat{y} \leq \theta$, *then* $\lambda_t(\boldsymbol{y}, \boldsymbol{d}) < \theta$.

Proof: (a) Since it follows from Lemma 5.1(a,b) that $v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1})$ is continuous in $\boldsymbol{x}_i$ and nondecreasing in $\boldsymbol{x}_i$ and $\boldsymbol{d}$, so also is $g_t^i(\xi; \boldsymbol{x}_i, \boldsymbol{d})$. Hence, since $g_t^i(\xi; \boldsymbol{x}_i, \boldsymbol{d})$ is strictly decreasing in ξ as seen in the proof of Lemma 5.3(a), we know $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$ is nondecreasing in $\boldsymbol{x}_i$ and $\boldsymbol{d}$.

Since it follows from Lemma 5.1(a,b) that $v_{t-1}((w, \boldsymbol{y}); (l, \boldsymbol{c}) - \mathbf{1})$ is continuous in $\boldsymbol{y}$ and nondecreasing in $\boldsymbol{y}$ and $\boldsymbol{d}$ for any w and l. Hence, if $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$ and $\boldsymbol{d}^1 \leq \boldsymbol{d}^2$, we obtain $f_t(\xi; \boldsymbol{y}^1, \boldsymbol{d}^1) = \max_{l \in K}\{-r(\xi, l) - s + \beta v_{t-1}((\xi, \boldsymbol{y}^1); (l, \boldsymbol{c}^1) - \mathbf{1})\} - \xi \leq \max_{l \in K}\{-r(\xi, l) - s + \beta v_{t-1}((\xi, \boldsymbol{y}^2); (l, \boldsymbol{c}^2) - \mathbf{1})\} - \xi = f_t(\xi; \boldsymbol{y}^2, \boldsymbol{d}^2)$, which shows that $f_t(\xi; \boldsymbol{y}, \boldsymbol{d})$ is nondecreasing in $\boldsymbol{y}$ and $\boldsymbol{d}$. Thereby, since it is strictly decreasing in ξ, we find that $\lambda_t(\boldsymbol{y}, \boldsymbol{d})$ is nondecreasing in $\boldsymbol{y}$ and $\boldsymbol{d}$.

(b) By definition of duration vector $\boldsymbol{d}$, it always holds $d_k \leq 0$. So, hereafter, we restrict i to $i < k$. Given any $i < k$, choose $\boldsymbol{d} \in \mathbb{R}^k$ and $\boldsymbol{x} \in \mathbb{R}^k$ so that $d_i \geq 1$, $x_i = \theta$, and $\hat{x}_i \leq \theta$. Note here that $\hat{x}_i = \max\{\delta_j x_j \mid j \neq i\}$ and that $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$ exists uniquely due to Lemma 5.3(a).

Let $\boldsymbol{p} = (0, \boldsymbol{y}) \in \mathbb{R}^k$ and $\boldsymbol{q} = (0, \boldsymbol{c}) - \mathbf{1} \in \mathbb{R}^k$, and define $\delta'_j = 1$ if $q_j \geq 0$ and $\delta'_j = 0$ if $q_j < 0$. Then, $p_{i+1} = x_i = \theta$, and $q_{i+1} = d_i - 1 \geq 0$ thus $\delta'_{i+1} = 1$. By definition of $\boldsymbol{q}$ we have $\delta'_j \leq \delta_{j-1}$ for each j. From these, especially by noting $p_1 = 0$ and $p_j = x_{j-1}$, we deduce

$$\hat{p}_{i+1} = \max\{\delta'_j p_j \mid j \neq i+1\} \leq \max\{\delta_{j-1} x_{j-1} \mid j \neq i+1\} = \max\{\delta_j x_j \mid j \neq i, k\} \leq \hat{x}_i \leq \theta,$$

implying $\hat{p} = \max\{1 \times p_{i+1}, \hat{p}_{i+1}\} = \theta$. As a result, due to Theorem 5.2(a) we obtain

$$v_{t-1}(\boldsymbol{p}; \boldsymbol{q}) = v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1}) = \psi(\theta). \tag{5.5}$$

By noting that a vector $\boldsymbol{x} = (x_1, \cdots, x_{i-1}, \theta, x_{i+1}, \cdots, x_k)$ is used to calculate $g_t^i(\theta; \boldsymbol{x}_i, \boldsymbol{d})$, it follows from (4.15) and (5.5) that $g_t^i(\theta; \boldsymbol{x}_i, \boldsymbol{d}) = -s + \beta\psi(\theta) - 1 \times \theta = \rho(\theta) = 0$. Hence, the uniqueness of $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d})$ leads us to conclude $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) = \theta$.

(c) Suppose $i = k$. Since $\boldsymbol{y}$ $(= \boldsymbol{x}_k)$ is independent of x_k, so also is $v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1})$. Hence, by (4.15) we get $\theta_t^k(\boldsymbol{x}_k, \boldsymbol{d}) = -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1})$. If $\boldsymbol{x}$ is such that $\hat{x}_k \leq \theta$, then $\hat{y} \leq \theta$, thus it follows from Lemma 5.1(b,a) and Theorem 5.2(a) that $v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1}) \leq v_{t-1}((0, \boldsymbol{y}); (-1, \boldsymbol{k})) \leq v_{t-1}((0, \theta, \cdots, \theta); (-1, \boldsymbol{k})) = \psi(\theta)$ where $\boldsymbol{k} = (k, \cdots, k) \in \mathbb{R}^{k-1}$. As a consequence, $\theta_t^k(\boldsymbol{x}_k, \boldsymbol{d}) = -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1}) \leq -s + \beta\psi(\theta) = \rho(\theta) + \theta = \theta$.

Suppose $i < k$. Let $\boldsymbol{x} \in \mathbb{R}^k$ and $\boldsymbol{d} \in \mathbb{R}^k$ denote those in the proof of (b), respectively, and choose $\boldsymbol{d}' \in \mathbb{R}^k$ so that $d'_i = 0$ and $d'_j = d_j$ for $j \neq i$. Since $\boldsymbol{d}' \leq \boldsymbol{d}$ due to $d'_i < 1 \leq d_i$, we get $v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}') - \mathbf{1}) \leq v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \mathbf{1}) = \psi(\theta)$ from Lemma 5.1(b) and (5.5). Hence, $g_t^i(\theta; \boldsymbol{x}_i, \boldsymbol{d}') \leq -s + \beta\psi(\theta) - 1 \times \theta = 0$. Since $g_t^i(\xi; \boldsymbol{x}_i, \boldsymbol{d}')$ is strictly decreasing in ξ, we conclude that $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}')$ must satisfy $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}') \leq \theta$ for $i < k$. We have thus completed the proof.

(d) Suppose $\boldsymbol{y} \in \mathbb{R}^{k-1}$ satisfies $\hat{y} \leq \theta$. Then, by Theorem 5.2(a) we get $v_{t-1}((\theta, \boldsymbol{y}); (l, \boldsymbol{c}) - \mathbf{1}) = \psi(\theta)$ for any $l \in K$, thus $f_t(\theta; \boldsymbol{y}, \boldsymbol{d}) = -r(\theta, \eta^*) + \rho(\theta) = -r(\theta, \eta^*) < 0$ where $\eta^* = \eta_t(\theta, \boldsymbol{y}, \boldsymbol{d})$. Hence, since $f_t(\xi; \boldsymbol{y}, \boldsymbol{d})$ is strictly decreasing in ξ, we claim $\lambda_t(\boldsymbol{y}, \boldsymbol{d})$ must satisfy $\lambda_t(\boldsymbol{y}, \boldsymbol{d}) < \theta$. ∎

Here, let us define the following four vectors:

$$\begin{aligned} \boldsymbol{d}_i^L &= (d_1, \cdots, d_i, -, \cdots, -) \in \mathbb{R}^k, & \boldsymbol{d}_i^R &= (-, \cdots, -, d_{i+1}, \cdots, d_k) \in \mathbb{R}^k, \\ \boldsymbol{c}_i^L &= (d_1, \cdots, d_i, -, \cdots, -) \in \mathbb{R}^{k-1}, & \boldsymbol{c}_i^R &= (-, \cdots, -, d_{i+1}, \cdots, d_{k-1}) \in \mathbb{R}^{k-1}. \end{aligned}$$

where "$-$" represents any negative integer. By definition of δ_i's, for example, $\boldsymbol{d}^1 = (1, -3, 2, 4, -1)$ and $\boldsymbol{d}^2 = (1, -1, 2, 4, -6)$ perform completely the same role. So, we need not to distinguish elements of $\boldsymbol{d}$ as long as they are negative. Defining $\hat{x}_i^L = \max\{\delta_1 x_1, \delta_2 x_2, \cdots, \delta_i x_i\}$ and $\hat{x}_i^R = \max\{\delta_{i+1} x_{i+1}, \delta_{i+2} x_{i+2}, \cdots, \delta_k x_k\}$, we arrive at

$$\hat{x} = \max\{\hat{x}_i^L,\ \hat{x}_i^R\}, \quad i \leq k. \tag{5.6}$$

Lemma 5.6

(a) *Suppose for a certain t and each i that if $v_{t-1}(\boldsymbol{x}; \boldsymbol{d}_i^R) < v_{t-1}(\boldsymbol{x}; \boldsymbol{d})$, then $v_{t-1}(\boldsymbol{x}; \boldsymbol{d}_i^L) = v_{t-1}(\boldsymbol{x}; \boldsymbol{d})$. Then, given any $\boldsymbol{x}^1$, $\boldsymbol{x}^2$, $\boldsymbol{d}^1$ and $\boldsymbol{d}^2$ such that $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$ and $\boldsymbol{d}^1 \leq \boldsymbol{d}^2$, we have $v_t(\boldsymbol{x}^1; \boldsymbol{d}^1) = v_t(\boldsymbol{x}^2; \boldsymbol{d}^2)$ if and only if $z_t^o(\boldsymbol{x}^1; \boldsymbol{d}^1) = z_t^o(\boldsymbol{x}^2; \boldsymbol{d}^2)$.*

(b) *For each t and i, if $v_t(\boldsymbol{x}; \boldsymbol{d}_i^R) < v_t(\boldsymbol{x}; \boldsymbol{d})$, then $v_t(\boldsymbol{x}; \boldsymbol{d}_i^L) = v_t(\boldsymbol{x}; \boldsymbol{d})$.*

Proof: (a) Choose $\boldsymbol{x}^1$, $\boldsymbol{x}^2$, $\boldsymbol{d}^1$ and $\boldsymbol{d}^2$ so that $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$ and $\boldsymbol{d}^1 \leq \boldsymbol{d}^2$. Let us consider three cases: (i) $\lambda_t(\boldsymbol{y}^2, \boldsymbol{d}^2) \leq w$, (ii) $w \notin W_t(\boldsymbol{x}^2, \boldsymbol{d}^2)$, and (iii) $w < \lambda_t(\boldsymbol{y}^2, \boldsymbol{d}^2)$ and $w \in W_t(\boldsymbol{x}^2, \boldsymbol{d}^2)$.

(i) In the case of $\lambda_t(\boldsymbol{y}^2, \boldsymbol{d}^2) \leq w$, since $\lambda_t(\boldsymbol{y}^1, \boldsymbol{d}^1) \leq w$ holds due to Theorem 5.5(a), it follows from (4.10) and Corollary 5.4(b2,b3) that $z_t^r((w, \boldsymbol{y}^1); \boldsymbol{d}^1) = z_t^r((w, \boldsymbol{y}^2); \boldsymbol{d}^2) = w$.

(ii) We easily get $z_t^r((w, \boldsymbol{y}^1); \boldsymbol{d}^1) \leq z_t^r((w, \boldsymbol{y}^2); \boldsymbol{d}^2)$ for any w. Hence, if $w \notin W_t(\boldsymbol{x}^2, \boldsymbol{d}^2)$, by (4.11) we deduce $z_t^r((w, \boldsymbol{y}^1); \boldsymbol{d}^1) \leq z_t^r((w, \boldsymbol{y}^2); \boldsymbol{d}^2) < z_t^o(\boldsymbol{x}^2; \boldsymbol{d}^2)$.

(iii) Here, define η^2 as

$$\eta^2 = \arg\max_{l\in K}\{-r(w,l) - s + \beta v_{t-1}((w,\boldsymbol{y}^2);(l,\boldsymbol{c}^2)-\mathbf{1})\}. \tag{5.7}$$

Suppose $w < \lambda_t(\boldsymbol{y}^2,\boldsymbol{d}^2)$ and $w \in W_t(\boldsymbol{x}^2,\boldsymbol{d}^2)$. Then, it follows from (4.9) to (4.11) and Corollary 5.4(b1) that $-s+\beta v_{t-1}((0,\boldsymbol{y}^2);(0,\boldsymbol{c}^2)-\mathbf{1}) \le z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2) \le z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2) = -r(w,\eta^2) - s + \beta v_{t-1}((w,\boldsymbol{y}^2);(\eta^2,\boldsymbol{c}^2)-\mathbf{1})$, which and the positivity of r provide $v_{t-1}((0,\boldsymbol{y}^2);(0,\boldsymbol{c}^2)-\mathbf{1}) < v_{t-1}((w,\boldsymbol{y}^2);(\eta^2,\boldsymbol{c}^2)-\mathbf{1})$. Hence, by Lemmas 4.1 and 5.1(b) we get, for any l with $l \ge \eta^2$,

$$v_{t-1}((w,\boldsymbol{y}^2);(0,\boldsymbol{c}^2)-\mathbf{1}) = v_{t-1}((0,\boldsymbol{y}^2);(0,\boldsymbol{c}^2)-\mathbf{1}) < v_{t-1}((w,\boldsymbol{y}^2);(l,\boldsymbol{c}^2)-\mathbf{1}). \tag{5.8}$$

In general, considering $i=1$, $\boldsymbol{x}=(w,\boldsymbol{y})$ and $\boldsymbol{d}=(l,\boldsymbol{c})-\mathbf{1}$ in our assumption, we can state that if $v_{t-1}((w,\boldsymbol{y});(0,\boldsymbol{c})-\mathbf{1}) < v_{t-1}((w,\boldsymbol{y});(l,\boldsymbol{c})-\mathbf{1})$, then $v_{t-1}((w,\boldsymbol{y});(l,\boldsymbol{c})-\mathbf{1}) = v_{t-1}((w,\boldsymbol{y});(l,\mathbf{0})-\mathbf{1})$. Hence, from (5.8) we have, for any l with $l \ge \eta^2$,

$$v_{t-1}((w,\boldsymbol{y}^2);(l,\boldsymbol{c}^2)-\mathbf{1}) = v_{t-1}((w,\boldsymbol{y}^2);(l,\mathbf{0})-\mathbf{1}). \tag{5.9}$$

Lemma 5.1(a,b) ensures, for any l,

$$v_{t-1}((w,\boldsymbol{y}^1);(l,\mathbf{0})-\mathbf{1}) \le v_{t-1}((w,\boldsymbol{y}^1);(l,\boldsymbol{c}^1)-\mathbf{1}) \le v_{t-1}((w,\boldsymbol{y}^2);(l,\boldsymbol{c}^2)-\mathbf{1}). \tag{5.10}$$

From Lemma 4.1, for any l,

$$v_{t-1}((w,\boldsymbol{y}^2);(l,\mathbf{0})-\mathbf{1}) = v_{t-1}((w,\boldsymbol{y}^1);(l,\mathbf{0})-\mathbf{1}). \tag{5.11}$$

By (5.9) to (5.11), for any l with $l \ge \eta^2$ we get $v_{t-1}((w,\boldsymbol{y}^1);(l,\boldsymbol{c}^1)-\mathbf{1}) = v_{t-1}((w,\boldsymbol{y}^2);(l,\boldsymbol{c}^2)-\mathbf{1})$, thus

$$-r(w,l) - s + \beta v_{t-1}((w,\boldsymbol{y}^1);(l,\boldsymbol{c}^1)-\mathbf{1}) = -r(w,l) - s + \beta v_{t-1}((w,\boldsymbol{y}^2);(l,\boldsymbol{c}^2)-\mathbf{1}). \tag{5.12}$$

It generally follows from Lemma 5.1(a,b) that

$$\max_{l\in K}\{-r(w,l) - s + \beta v_{t-1}((w,\boldsymbol{y}^1);(l,\boldsymbol{c}^1)-\mathbf{1})\} \le \max_{l\in K}\{-r(w,l) - s + \beta v_{t-1}((w,\boldsymbol{y}^2);(l,\boldsymbol{c}^2)-\mathbf{1})\}.$$

From this, (5.7) and (5.12), we get $\max_l\{-r(w,l)-s+\beta v_{t-1}((w,\boldsymbol{y}^1);(l,\boldsymbol{c}^1)-\mathbf{1})\} = \max_l\{-r(w,l)-s+\beta v_{t-1}((w,\boldsymbol{y}^2);(l,\boldsymbol{c}^2)-\mathbf{1})\}$, thus $z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1) = z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2)$.

Owing to parts (i) to (iii), under the assumption it follows for any w that either $z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1) = z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2)$ holds or $z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1) \le z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2) < z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2)$ holds.

To finish the proof we shall examine two cases. First, suppose $z_t^o(\boldsymbol{x}^1;\boldsymbol{d}^1) = z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2)$. Then, if w satisfies $z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1) = z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2)$, clearly $\max\{z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1), z_t^o(\boldsymbol{x}^1;\boldsymbol{d}^1)\} = \max\{z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2), z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2)\}$. On the contrary, if w satisfies $z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1) \ne z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2)$, since $z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1) < z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2) < z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2) = z_t^o(\boldsymbol{x}^1;\boldsymbol{d}^1)$ holds in this case, we get $\max\{z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1), z_t^o(\boldsymbol{x}^1;\boldsymbol{d}^1)\} = z_t^o(\boldsymbol{x}^1;\boldsymbol{d}^1) = z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2) = \max\{z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2), z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2)\}$. Hence, for any w, if $z_t^o(\boldsymbol{x}^1;\boldsymbol{d}^1) = z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2)$, then $v_t(\boldsymbol{x}^1;\boldsymbol{d}^1) = v_t(\boldsymbol{x}^2;\boldsymbol{d}^2)$ from (4.13).

Conversely, suppose $z_t^o(\boldsymbol{x}^1;\boldsymbol{d}^1) < z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2)$. As seen in above, for any $w \notin W_t(\boldsymbol{x}^2,\boldsymbol{d}^2)$ we have $z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1) \le z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2) < z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2)$, from which $\max\{z_t^r((w,\boldsymbol{y}^1);\boldsymbol{d}^1), z_t^o(\boldsymbol{x}^1;\boldsymbol{d}^1)\} < z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2) = \max\{z_t^r((w,\boldsymbol{y}^2);\boldsymbol{d}^2), z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2)\}$. Thereby, we arrive at if $z_t^o(\boldsymbol{x}^1;\boldsymbol{d}^1) < z_t^o(\boldsymbol{x}^2;\boldsymbol{d}^2)$, then $v_t(\boldsymbol{x}^1;\boldsymbol{d}^1) < v_t(\boldsymbol{x}^2;\boldsymbol{d}^2)$.

By adding the results in two cases above, we have verified the truth of this assertion.

(b) We should note first that if $i = k$, then $\boldsymbol{d}_k^L = \boldsymbol{d}$, thus $v_t(\boldsymbol{x};\boldsymbol{d}_i^L) = v_t(\boldsymbol{x};\boldsymbol{d})$, which holds true whether $v_t(\boldsymbol{x};\boldsymbol{d}_i^R) < v_t(\boldsymbol{x};\boldsymbol{d})$ or not.

The proof for $i < k$ is done by induction on t. For $t = 0$, it follows from (3.1) that $v_0(\boldsymbol{x}; \boldsymbol{d}_i^L) = \psi(\hat{x}_i^L)$, $v_0(\boldsymbol{x}; \boldsymbol{d}_i^R) = \psi(\hat{x}_i^R)$ and $v_0(\boldsymbol{x}; \boldsymbol{d}) = \psi(\hat{x})$. Hence, due to Lemma 3.1(b) we find that if $v_0(\boldsymbol{x}; \boldsymbol{d}_i^R) < v_0(\boldsymbol{x}; \boldsymbol{d})$, then $\hat{x}_i^R < \hat{x}$ and $a < \hat{x}$. Since $\hat{x}_i^R < \hat{x}$ implies $\hat{x}_i^L = \hat{x}$ due to (5.6), we arrive at $v_0(\boldsymbol{x}; \boldsymbol{d}_i^L) = \psi(\hat{x}_i^L) = \psi(\hat{x}) = v_0(\boldsymbol{x}; \boldsymbol{d})$, which assures the assertion to be true for $t = 0$.

Now, suppose for each i that if $v_{t-1}(\boldsymbol{x}; \boldsymbol{d}_i^R) < v_{t-1}(\boldsymbol{x}; \boldsymbol{d})$, then $v_{t-1}(\boldsymbol{x}; \boldsymbol{d}_i^L) = v_{t-1}(\boldsymbol{x}; \boldsymbol{d})$. To complete the proof, owing to assertion (a) it suffices to show that if $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^R) < z_t^o(\boldsymbol{x}; \boldsymbol{d})$, then $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^L) = z_t^o(\boldsymbol{x}; \boldsymbol{d})$, where $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^R) = \max\{\hat{x}_i^R, -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}_i^R) - \boldsymbol{1})\}$ and $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^L) = \max\{\hat{x}_i^L, -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}_i^L) - \boldsymbol{1})\}$.

In the case of $z_t^o(\boldsymbol{x}; \boldsymbol{d}) = \hat{x}$, inequality $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^R) < z_t^o(\boldsymbol{x}; \boldsymbol{d})$ implies $\hat{x}_i^R < \hat{x}$. This and (5.6) yield $\hat{x}_i^L = \hat{x}$. Lemma 5.1(b) indicates $v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}_i^L) - \boldsymbol{1}) \le v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \boldsymbol{1})$. Hence, in this case, if $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^R) < z_t^o(\boldsymbol{x}; \boldsymbol{d})$, then $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^L) = \hat{x}_i^L = \hat{x} = z_t^o(\boldsymbol{x}; \boldsymbol{d})$.

In the case of $z_t^o(\boldsymbol{x}; \boldsymbol{d}) = -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \boldsymbol{1})$, inequality $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^R) < z_t^o(\boldsymbol{x}; \boldsymbol{d})$ implies $-s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}_i^R) - \boldsymbol{1}) < -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \boldsymbol{1})$, or equivalently,

$$v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}_i^R) - \boldsymbol{1}) < v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \boldsymbol{1}). \tag{5.13}$$

By considering $\boldsymbol{d} = (0, \boldsymbol{c}) - \boldsymbol{1}$ we get $\boldsymbol{d}_{i+1}^R = (0, \boldsymbol{c}_i^R) - \boldsymbol{1}$ and $\boldsymbol{d}_{i+1}^L = (0, \boldsymbol{c}_i^L) - \boldsymbol{1}$. Accordingly, by our assumption, (5.13) yields $v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}_i^L) - \boldsymbol{1}) = v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \boldsymbol{1})$. Hence, in this case, if $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^R) < z_t^o(\boldsymbol{x}; \boldsymbol{d})$, then $\hat{x}_i^L \le \hat{x} \le -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}) - \boldsymbol{1}) = -s + \beta v_{t-1}((0, \boldsymbol{y}); (0, \boldsymbol{c}_i^L) - \boldsymbol{1})$, thus $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^L) = z_t^o(\boldsymbol{x}; \boldsymbol{d})$.

Now, under the assumption we have confirmed that if $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^R) < z_t^o(\boldsymbol{x}; \boldsymbol{d})$, then $z_t^o(\boldsymbol{x}; \boldsymbol{d}_i^L) = z_t^o(\boldsymbol{x}; \boldsymbol{d})$, or equivalently, if $v_t(\boldsymbol{x}; \boldsymbol{d}_i^R) < v_t(\boldsymbol{x}; \boldsymbol{d})$, then $v_t(\boldsymbol{x}; \boldsymbol{d}_i^L) = v_t(\boldsymbol{x}; \boldsymbol{d})$ due to assertion (a). As a result the proof is completed. ∎

Corollary 5.7

(a) *For each t and i we have* $\max\{v_t(\boldsymbol{x}; \boldsymbol{d}_i^L), v_t(\boldsymbol{x}; \boldsymbol{d}_i^R)\} = v_t(\boldsymbol{x}; \boldsymbol{d})$.

(b) *If $\boldsymbol{x}^1 \le \boldsymbol{x}^2$ and $\boldsymbol{d}^1 \le \boldsymbol{d}^2$, for any $w \in W_t(\boldsymbol{x}^2, \boldsymbol{d}^2)$ we have* $z_t^r((w, \boldsymbol{y}^1); \boldsymbol{d}^1) = z_t^r((w, \boldsymbol{y}^2); \boldsymbol{d}^2)$.

Proof: (a) Immediate due to Lemma 5.6(a,b).

(b) Note first that (a) assures the validity of the premise of Lemma 5.6(a) for each t.

If $t = 0$, by definition we conclude $z_0^r((w, \boldsymbol{y}^1); \boldsymbol{d}^1) = z_0^r((w, \boldsymbol{y}^2); \boldsymbol{d}^2) = w$ for any w.

Let $t > 0$ and $w \in W_t(\boldsymbol{x}^2, \boldsymbol{d}^2)$. If w also satisfies $w < \lambda_t(\boldsymbol{y}^2, \boldsymbol{d}^2)$, we obtain $z_t^r((w, \boldsymbol{y}^1); \boldsymbol{d}^1) = z_t^r((w, \boldsymbol{y}^2); \boldsymbol{d}^2)$ in exactly the same manner as in part (iii) of the proof of Lemma 5.6(a). Conversely, if $\lambda_t(\boldsymbol{y}^2, \boldsymbol{d}^2) \le w$, see part (i) of the proof of Lemma 5.6(a) and we get $z_t^r((w, \boldsymbol{y}^1); \boldsymbol{d}^1) = z_t^r((w, \boldsymbol{y}^2); \boldsymbol{d}^2) = w$. As a result the assertion proves true. ∎

Theorem 5.8

(a) *If $\boldsymbol{x}^1 \le \boldsymbol{x}^2$ and $\boldsymbol{d}^1 \le \boldsymbol{d}^2$, then* $W_t(\boldsymbol{x}^1, \boldsymbol{d}^1) \supseteq W_t(\boldsymbol{x}^2, \boldsymbol{d}^2)$.

(b) *For any $w \in W_t(\boldsymbol{x}, \boldsymbol{d})$ we have $w \le \lambda_t(\boldsymbol{y}, \boldsymbol{d})$ if and only if $w \le \lambda_t(\boldsymbol{0}, -)$.*

Proof: (a) Choose $\boldsymbol{x}^1$, $\boldsymbol{x}^2$, $\boldsymbol{d}^1$ and $\boldsymbol{d}^2$ so that $\boldsymbol{x}^1 \le \boldsymbol{x}^2$ and $\boldsymbol{d}^1 \le \boldsymbol{d}^2$. Easily we find $z_t^o(\boldsymbol{x}^1; \boldsymbol{d}^1) \le z_t^o(\boldsymbol{x}^2; \boldsymbol{d}^2)$. Hence, it follows from (4.11) and Corollary 5.7(b) that if $w \in W_t(\boldsymbol{x}^2, \boldsymbol{d}^2)$, then $z_t^o(\boldsymbol{x}^1; \boldsymbol{d}^1) \le z_t^o(\boldsymbol{x}^2; \boldsymbol{d}^2) \le z_t^r((w, \boldsymbol{y}^2); \boldsymbol{d}^2) = z_t^r((w, \boldsymbol{y}^1); \boldsymbol{d}^1)$, thus $w \in W_t(\boldsymbol{x}^1, \boldsymbol{d}^1)$, implying $W_t(\boldsymbol{x}^1, \boldsymbol{d}^1) \supseteq W_t(\boldsymbol{x}^2, \boldsymbol{d}^2)$.

(b) We here use λ^y and λ^0 instead of $\lambda_t(\boldsymbol{y}, \boldsymbol{d})$ and $\lambda_t(\boldsymbol{0}, -)$, respectively.

First we shall show that if $w \in W_t(\boldsymbol{x}, \boldsymbol{d})$ and $w = \lambda^y$, then $w = \lambda^0$. Suppose $\lambda^y \in W_t(\boldsymbol{x}, \boldsymbol{d})$. Then, imitating the argument in part (iii) of the proof of Lemma 5.6(a), we arrive at $\max_l\{-r(\lambda^y, l) - s + \beta v_{t-1}((\lambda^y, \boldsymbol{0}); (l, -) - \boldsymbol{1})\} = \max_l\{-r(\lambda^y, l) - s + \beta v_{t-1}((\lambda^y, \boldsymbol{y}); (l, \boldsymbol{c}) - \boldsymbol{1})\}$, thus due to (4.16) we get $f_t(\lambda^y; \boldsymbol{0}, -) = f_t(\lambda^y; \boldsymbol{y}, \boldsymbol{d})$. By definition of λ^y we have $f_t(\lambda^y; \boldsymbol{y}, \boldsymbol{d}) = 0$, thus

$f_t(\lambda^y; \mathbf{0}, -) = 0$, which indicates $\lambda^y = \lambda^0$ according to Lemma 5.3(b). Thereby, if $\lambda^y \in W_t(\boldsymbol{x}, \boldsymbol{d})$, then $\lambda^y = \lambda^0$, that is, if $w \in W_t(\boldsymbol{x}, \boldsymbol{d})$ and $w = \lambda^y$, then $w = \lambda^0$.

Next, we shall show that if $w \in W_t(\boldsymbol{x}, \boldsymbol{d})$ and $w < \lambda^y$, then $w < \lambda^0$. Suppose that a certain $w \in W_t(\boldsymbol{x}, \boldsymbol{d})$ satisfies $\lambda^0 \leq w < \lambda^y$. Then, due to Corollary 5.4(b) we have $z_t^r((w, \mathbf{0}); -) = w < \max_l\{-r(w,l) - s + \beta v_{t-1}((w, \boldsymbol{y}); (l, \boldsymbol{c}) - \mathbf{1})\} = z_t^r((w, \boldsymbol{y}); \boldsymbol{d})$, which contradicts Corollary 5.7(b). As a result, it follows for any $w \in W_t(\boldsymbol{x}, \boldsymbol{d})$ that not $\lambda^0 \leq w < \lambda^y$, that is, if $w \in W_t(\boldsymbol{x}, \boldsymbol{d})$ and $w < \lambda^y$, then $w < \lambda^0$. This and the previous fact show that if $w \in W_t(\boldsymbol{x}, \boldsymbol{d})$ and $w \leq \lambda^y$, then $w \leq \lambda^0$.

Conversely, it follows from Theorem 5.5(a) that if $w \leq \lambda^0$, then $w \leq \lambda^y$. We have thus confirmed the assertion. ∎

6. Conclusions

We here summarize four properties of the optimal decision rule.

1. If the leading offer $\hat{x}$ is such as $\theta \leq \hat{x}$, then you should instantly stop the search process by accepting the more lucrative one between current offer w and leading offer $\hat{x}$.

This is already stated as Rule II. As seen in the next result, however, there exists no chance to satisfy $\theta \leq \hat{x}$ except for the special case that such an $\boldsymbol{x}$ is given as an initial offer before entering the search process.

2. An offer reserved during the search process must not be accepted prior to its maturity of reservation, however, it may be accepted on the maturity.

If $\boldsymbol{x}$ and $\boldsymbol{d}$ are such that $\hat{x} < \theta$, then $\hat{y} < \theta$, thus $\lambda_t(\boldsymbol{y}, \boldsymbol{d}) < \theta$ for each t due to Theorem 5.5(d). From this and Rule III we find that all offers to be reserved have less value than θ throughout the search process. Accordingly, if the search starts with $\boldsymbol{x}$ and $\boldsymbol{d}$ such that $\hat{x} < \theta$, the inequality holds forever, or $x_i < \theta$ for all i for every $t > 0$. While $d_i \geq 1$, we never have $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) < x_i$ since $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) = \theta$ by Theorem 5.5(b). Hence, if x_i is the leading offer with $d_i \geq 1$, then PS is not the optimal decision. If $d_i = 0$, since $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) \leq \theta$, it is possible to have a reserved offer x_i satisfying $\theta_t^i(\boldsymbol{x}_i, \boldsymbol{d}) < x_i$. Hence, if x_i is the leading offer with $d_i = 0$, then PS may be the optimal decision.

In the model, every time you decide to proceed the search once more, a positive search cost must be spent and chances to meet offers superior to reserved ones up to the deadline decrease. So, sometime we reach the time when we begin to feel wasteful for pursuing the search further, and this seems to be the time we should stop the search by accepting either the leading offer or the current offer. Even if we are on that time, however, the second property suggests us not to recall the leading offer as long as its reserving duration has not been exhausted. One explanation for why we should refrain from recalling any reserved offer prior to its maturity may be that although the purpose of reserving offers is to facilitate stopping the process when we see no reason to continue it further, we feel more waste to recall an offer while it remains available at the next time also, than to pay search costs and reduce the probability of finding better offers.

3. If $\boldsymbol{x}$ or $\boldsymbol{d}$ is better, the range of offers to be passed up should be wider.

Theorem 5.8(a) indicates this result, which is intuitively clear because if $\boldsymbol{x}$ or $\boldsymbol{d}$ is better, we feel less need of providing new recallable offers.

4. For each time t with any $\boldsymbol{x}$ and $\boldsymbol{d}$, an offer w to be reserved satisfies $w \in W_t(\boldsymbol{x})$ and $w \leq \lambda_t(\mathbf{0}, -)$.

This is derived from Theorem 5.8(b) and suggests that whatever $\boldsymbol{x}$ we have, it suffices to know $\lambda_t(\mathbf{0}, -)$ so as to determine whether to accept or reserve an offer.

These four properties are found also in Saito [10] and quite similar ones are in Saito [9]. This fact leads us to conclude that these properties are essential for the optimal decision rule to optimal stopping problems with reservation where reserving costs are incurred to reserve offers, whether the term of validity of reservation is fixed or not.

One of the subjects to be examined is to reveal properties of $\eta_t(w, \boldsymbol{y}, \boldsymbol{d})$ because they are left untouched in this paper although the current model assumes the remaining duration to be determinable.

References

[1] Ikuta, S. Optimal stopping problem with uncertain recall. *Journal of the Operations Research Society of Japan*, 31(2):145–170, 1988.

[2] Ikuta, S. Optimal stopping problem with several search areas. *Journal of the Operations Research Society of Japan*, 38(1):89–106, 1995.

[3] Ikuta, S. *K-function and optimal stopping problems.* Discussion paper, No.705, University of Tsukuba, 1996.

[4] Lippman, S. A. and McCardle, K. F. Uncertain search: a model of search among technologies of uncertain values. *Management Science*, 37(11):1474–1490, 1991.

[5] Morgan, P. and Manning, R. Optimal search. *Econometrica*, 53(4):923–944, 1985.

[6] Rose, J. S. Optimal sequential selection based on relative ranks with renewable call options. *Journal of the American Statistical Association*, 79(386):430–435, 1984.

[7] Rose, J. S. The secretary problem with a call option. *Operations Research Letters*, 3(5):237–241, 1984.

[8] Ross, S. M. A problem in optimal search and stop. *Operations Research*, 17:984–992, 1969.

[9] Saito, T. Optimal stopping problem with controlled recall. *Probability in the Engineering and Informational Sciences*, 12(1):91–108, 1998.

[10] Saito, T. Optimal stopping problem with finite-period reservation. *European Journal of Operational Research*, 118:605–619, 1999.

[11] Zuckerman, D. On preserving the reservation wage property in a continuous job search model. *Journal of Economic Theory*, 34(1):175–179, 1984.